Mathematical Methods Exam 1: Solutions

Question 1

a. $f(x) = \dfrac{2}{x-3} + 4$

Let $y = \dfrac{2}{x-3} + 4$

Inverse: swap x and y.

$x = \dfrac{2}{y-3} + 4$ **1M**

$(x-4)(y-3) = 2$

$y = \dfrac{2}{x-4} + 3$ **1A**

$f^{-1}: R\backslash\{4\} \to R$, where

$f^{-1}(x) = \dfrac{2}{x-4} + 3$ **1A**

b. Equating f to f^{-1} or f to x,

$x = \dfrac{2}{x-3} + 4$ **1M**

(alternatively, $\dfrac{2}{x-3} + 4 = \dfrac{2}{x-4} + 3$)

$(x-4)(x-3) = 2$

$x^2 - 7x + 10 = 0$

$(x-5)(x-2) = 0$

$x = 5$ or $x = 2$

$(5, 5)$ and $(2, 2)$ **1A**

Question 2

$2\sin^2(x) = 1$

$\sin^2(x) = \dfrac{1}{2}$

$\sin(x) = \pm\dfrac{1}{\sqrt{2}}$ **1A**

$x = \dfrac{-3\pi}{4}, \dfrac{-\pi}{4}, \dfrac{\pi}{4}, \dfrac{3\pi}{4}$ **2A**

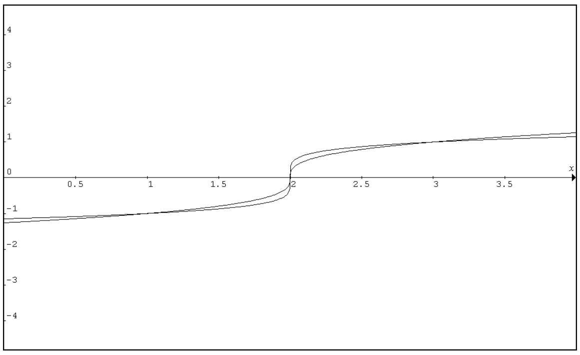
(i.e. 4 × ½A)

Question 3

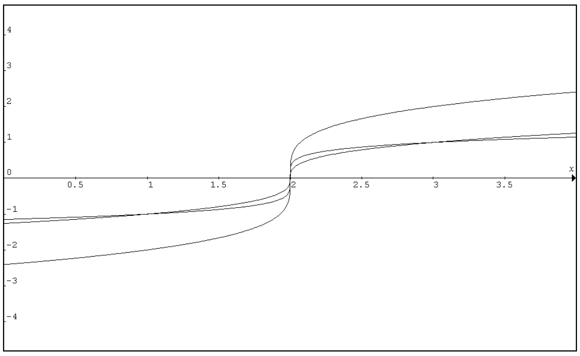
a. Correct Endpoints **1A**

Correct Intersections **1A**

Correct Shape **1A**



b. Correct shape and open circles for endpoints **1A**



c. $2\displaystyle\int_2^3 \left((x-2)^{\frac{1}{5}} - (x-2)^{\frac{1}{3}}\right)dx$ **1M**

$= 2\left[\dfrac{5}{6}(x-2)^{\frac{6}{5}} - \dfrac{3}{4}(x-2)^{\frac{4}{3}}\right]_2^3$ **1A**

$= 2\left(\left(\dfrac{5}{6}(3-2)^{\frac{6}{5}} - \dfrac{3}{4}(3-2)^{\frac{4}{3}}\right) - \left(\dfrac{5}{6}(2-2)^{\frac{6}{5}} - \dfrac{3}{4}(2-2)^{\frac{4}{3}}\right)\right)$

$= 2\left(\dfrac{5}{6} - \dfrac{3}{4}\right)$

$= \dfrac{1}{6}$ square units **1A**

Question 4

a. Rule: $f(g(x)) = \log_e(|2x+5|)$ **1A**

Domain: $R^- \setminus \{-\frac{5}{2}\}$ or $(-\infty, 0) \setminus \{-\frac{5}{2}\}$ **1A**

or

$f(g(x)): (-\infty, 0) \setminus \{-\frac{5}{2}\} \rightarrow R$, where

$f(g(x)) = \log_e(|2x+5|)$

b. $\frac{d}{dx}[f(g(x))] = \frac{2}{2x+5}$ **1M**

$\frac{d}{dx}[f(g(-4))] = \frac{2}{-8+5}$

$= \frac{-2}{3}$ **1M**

$f(g(-4)) = \log_e(3)$

$y - \log_e(3) = \frac{3}{2}(x+4)$

$y = \frac{3}{2}x + 6 + \log_e(3)$ **1A**

Question 5

a. $h = 2r$

$V = \pi r^2 h$

$= \frac{\pi h^3}{4}$ **1M**

b. $\frac{dV}{dh} = \frac{3\pi h^2}{4}$ **1M**

$\frac{dh}{dt} = \frac{dV}{dt}\frac{dh}{dv}$

$= 8 \times \frac{4}{3\pi h^2}$ **1M**

When $h = 2$ cm

$\frac{dh}{dt} = \frac{8}{3\pi}$ cm/s **1A**

Question 6

a. $f(x) = \frac{2e^{3x}}{\sqrt{x+1}}$ **1M**

$$f'(x) = \frac{6e^{3x}\sqrt{(x+1)} - \frac{1}{2\sqrt{(x+1)}}2e^{3x}}{x+1}$$

$= \frac{6e^{3x}(x+1) - e^{3x}}{(x+1)^{\frac{3}{2}}}$ **1M**

$= \frac{e^{3x}(6x+5)}{(x+1)^{\frac{3}{2}}}$ **1A**

b. $f'(x) = \frac{e^{3x}(6x+5)}{(x+1)^{\frac{3}{2}}} = 0$

$6x + 5 = 0$

$x = -\frac{5}{6}$ **1M**

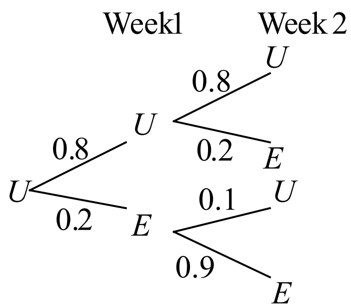
x	-0.9	$-\frac{5}{6}$	0
$f'(x)$	–ve	0	+ve

Local minimum stationary point at **1A**

$\left(-\frac{5}{6}, 2\sqrt{6}\,e^{-\frac{5}{2}}\right)$ **1A**

Question 7

Let U denote the purchase of unleaded and E denote the purchase of ethanol blend.



$\Pr(UE + EE)$ **1M**

$= 0.8 \times 0.2 + 0.2 \times 0.9$ **1M**

$= 0.16 + 0.18$

$= 0.34$ **1A**

Question 8

a. $\int_0^{\pi} \cos\left(\frac{x}{2}\right)dx$

$= \left[2\sin\left(\frac{x}{2}\right)\right]_0^{\pi}$ **1M**

$= 2\sin\left(\frac{\pi}{2}\right) - 0$

$= 2$ **1A**

b. $k\int_0^{\pi} \cos\left(\frac{x}{2}\right)dx = 1$

$2k = 1$

$k = \frac{1}{2}$ **1A**

c. $\frac{d}{dx}\left(x\sin\left(\frac{x}{2}\right) + 2\cos\left(\frac{x}{2}\right)\right)$

$= \sin\left(\frac{x}{2}\right) + \frac{1}{2}x\cos\left(\frac{x}{2}\right) - \sin\left(\frac{x}{2}\right)$ **1M**

$= \frac{1}{2}x\cos\left(\frac{x}{2}\right)$ **1A**

d. $\int_0^{\pi} x\cos\left(\frac{x}{2}\right)dx = 2\left[x\sin\left(\frac{x}{2}\right) + 2\cos\left(\frac{x}{2}\right)\right]_0^{\pi}$

$= 2(\pi - 2)$

$= 2\pi - 4$ **1A**

e. $\mathrm{E}(X) = \int_0^{\pi} \left(x \times \frac{1}{2}\cos\left(\frac{x}{2}\right)\right)dx$

$= \frac{1}{2}(2\pi - 4)$

$= \pi - 2$ **1A**